An Extremal Problem On Potentially $K_{r+1} - H$-graphic Sequences *

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Abstract

Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). A sequence S is potentially $K_m - H$-graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n-term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$-graphical. In this paper, we determine the values of $\sigma(K_{r+1} - H, n)$ for $n \geq 4r+10, r \geq 3, r+1 \geq k \geq 4$ where H is a graph on k vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices. We also determine the values of $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r+8, r \geq 3$.

Key words: graph; degree sequence; potentially $K_{r+1} - H$-graphic sequence
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1 Introduction

The set of all non-increasing nonnegative integers sequence $\pi = (d(v_1), d(v_2), ..., d(v_n))$ is denoted by NS_n. A sequence $\pi \epsilon NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and

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such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H-graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. Let $\sigma(\pi) = d(v_1)+d(v_2)+...+d(v_n)$, and $[x]$ denote the largest integer less than or equal to x. We denote $G + H$ as the graph with $V(G + H) = V(G) \bigcup V(H)$ and $E(G + H) = E(G) \bigcup E(H) \bigcup \{xy : x \in V(G), y \in V(H)\}$. Let K_k, C_k, T_k, and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k+1$ vertices, and a path on $k+1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m).

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [2] in 1938 and generalized by Turán [15]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphical. Here we consider the following variant: determine the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H-graphical. We denote this minimum l by $\sigma(H, n)$. Erdös, Jacobson and Lehel [4] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [5],[10],[11],[12] and [13].

Gould, Jacobson and Lehel [5] also proved that $\sigma(pK_2, n) = (p-1)(2n-2) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. Luo [14] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Lai [7] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Lai [8, 9] determined $\sigma(K_5 - C_4, n), \sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$, for $n \geq 5$. Yin, Li and Mao[17] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3$, $r+1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Yin and Li[16] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$. After reading[16], using Yin-Li method Yin[18] determined the values $\sigma(K_{r+1} - K_3, n)$ for $r \geq 3, n \geq 3r + 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading[16] and [18], using Yin-Li method we prove the following three theorems.

Theorem 1.1. If $r \geq 3$ and $n \geq 4r + 8$, then $\sigma(K_{r+1} - P_2, n) = (r-1)(2n-r) - 2(n-r) + 2$.

Theorem 1.2. If $r \geq 3$ and $n \geq 4r + 10$, then $\sigma(K_{r+1} - T_3, n) = (r-1)(2n-r) - 2(n-r)$.

Theorem 1.3. If $r \geq 3, r+1 \geq k \geq 4$ and $n \geq 4r+10$, then $\sigma(K_{r+1}-H, n) = (r-1)(2n-r) - 2(n-r)$, where H is a graph on k vertices which contains a tree on 4 vertices but not contains a cycle on 3 vertices.

There are a number of graphs on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices (for example, the cycle on k vertices, the tree on k vertices, and the complete 2-partite graph on k vertices, etc).

2 Preparations

In order to prove our main result,we need the following notations and results.

Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n, 1 \leq k \leq n$. Let

$$\pi''_k = \begin{cases} (d_1-1, \cdots, d_{k-1}-1, d_{k+1}-1, \cdots, d_{d_k+1}-1, d_{d_k+2}, \cdots, d_n), \\ \text{ if } d_k \geq k, \\ (d_1-1, \cdots, d_{d_k}-1, d_{d_k+1}, \cdots, d_{k-1}, d_{k+1}, \cdots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi'_k = (d'_1, d'_2, \cdots, d'_{n-1})$,where $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ is a rearrangement of the $n-1$ terms of π''_k. Then π'_k is called the residual sequence obtained by laying off d_k from π.

Theorem 2.1[16] Let $n \geq r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r+1} \geq r$. If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.2[16] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r+1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially A_{r+1}-graphic.

Theorem 2.3[16] Let $n \geq r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-1$, then π is potentially $K_{r+1}-e$-graphic.

Theorem 2.4[16] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r-1} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-e$ -graphic.

Theorem 2.5[6] Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n$ and $1 \leq k \leq n$. Then $\pi \epsilon GS_n$ if and only if $\pi'_k \epsilon GS_{n-1}$.

Theorem 2.6[3] Let $\pi = (d_1, \cdots, d_n) \epsilon NS_n$ with even $\sigma(\pi)$. Then $\pi \epsilon GS_n$ if and only if for any t,$1 \leq t \leq n-1$,

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{j=t+1}^{n} min\{t, d_j\}.$$

Theorem 2.7[5] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization

G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Lemma 2.1 **[18]** If $\pi = (d_1, d_2, \cdots, d_n) \epsilon NS_n$ is potentially $K_{r+1} - e$-graphic, then there is a realization G of π containing $K_{r+1} - e$ with the $r+1$ vertices $v_1, \cdots, v_{r+1}$ such that $d_G(v_i) = d_i$ for $i = 1, 2, \cdots, r+1$ and $e = v_r v_{r+1}$.

Lemma 2.2 **[18]** If $r \geq 3$ and $n \geq r+1$, then $\sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$.

3 Proof of Main results.

Lamma 3.1 Let $n \geq r+1$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_r \geq r-1$ and $d_{r+1} \geq r-2$. If $d_i \geq 2r - i$ for $i = 1, 2, \cdots, r-2$, then π is potentially $K_{r+1} - P_2$-graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r-1$.

Subcase 1.1: $d_{r-1} \geq r+1$. Then π is potentially $K_{r+1} - e$-graphic by Theorem 2.3. Hence,π is potentially $K_{r+1} - P_2$-graphic.

Subcase 1.2: $d_{r-1} = r-1$. Then $d_{r-1} = d_r = d_{r+1} = r-1$.

If $d_{r+2} = r-1$, then the residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$, (2) $d'_1 \geq 2(r-1) - 1, \cdots$, $d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{r-1} = d_r, d'_{(r-1)+1} = d'_r = d_{r+2} = r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r+2}\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_{r+2} \leq r-2$, then the residual sequence $\pi'_{r+1} = (d'_1, \cdots, d'_{n-1})$ obtained by laying off $d_{r+1} = r-1$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$, (2) $d'_1 \geq 2(r-1) - 1, \cdots, d'_{(r-1)-1} = d'_{r-2} \geq 2(r-1) - (r-2), d'_{r-1} = d_r$, $d'_{(r-1)+1} = d'_r = d_{r-1} - 1 = r-2$. By Theorem 2.3, π'_{r+1} is potentially $K_{(r-1)+1} - e$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r-1} - 1\} = \{d'_1, \cdots, d'_r\}$ and Lemma 2.1.

Subcase 1.3: $d_{r-1} = r$. Then $d_{r+1} = r$ or $r-1$.

If $d_{r+1} = r$, then $d_{r-1} = d_r = d_{r+1} = r$. The residual sequence π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \cdots$, $d'_{(r-1)-1} = d'_{r-2} \geq d_{r-2} - 1 \geq 2(r-1) - (r-2)$ and $d'_{(r-1)+1} = d'_r \geq d_r - 1 = r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Thus, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1\} \subseteq \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_{r+1} = r-1$, then $d_r = r-1$ or r.

If $d_r = r-1$, then π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-1$,(2) $d'_1 \geq d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r-1)-(r-2)$ and $d'_{(r-1)+1} = d'_r = d_r = r-1$. According to Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-1} - 1, d_r\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_r = r$, then π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 \geq d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r-1) - (r-2)$ and $d'_{(r-1)+1} = d'_r = d_{r-1} - 1 = r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r-1} - 1\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Case 2: $d_{r+1} \leq r-2$, that is, $d_{r+1} = r-2$.

If $d_{r-1} < d_{r-2}$, then π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_1 = d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-1} = d'_{r-2} = d_{r-2} - 1 \geq 2(r-1) - [(r-1)-1]$ and $d'_{(r-1)+1} = d'_r = d_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

If $d_{r-1} = d_{r-2} \geq r+2$, then π'_{r+1} satisfies: $d'_1 \geq d_1 - 1 \geq 2(r-1)-1, \cdots, d'_{(r-1)-1} = d'_{r-2} \geq d_{r-2} - 1 \geq 2(r-1) - [(r-1)-1]$ and $d'_{(r-1)+1} = d'_r \geq r-1$. By Theorem 2.1, π'_{r+1} is potentially $A_{(r-1)+1}$-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_{r-1}, d_r, d_1 - 1, \cdots, d_{r-2} - 1\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.2. Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$ with $d_{r-2} \geq r$. If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - P_2$-graphic.

Proof. We consider the following two cases.

Case 1: If $d_{r-1} \geq r$. Then π is potentially $K_{r+1} - e$-graphic by Theorem 2.4. Hence, π is potentially $K_{r+1} - P_2$-graphic.

Case 2: $d_{r-1} \leq r-1$, that is, $d_{r-1} = r-1$, then $d_{r-1} = d_r = d_{r+1} = \cdots = d_{2r+2} = r-1$ and π'_{r+1} satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \cdots, r-2$,(2) $d'_{(r-1)+1} = d'_r \geq r-1$ and $d'_{2(r-1)+2} = d'_{2r} \geq (r-1)-1$. By Theorem 2.2, π'_{r+1} is potentially A_r-graphic. Therefore, π is potentially $K_{r+1} - P_2$-graphic by $\{d_1 - 1, \cdots, d_{r-2} - 1, d_r, d_{r+2}\} = \{d'_1, \cdots, d'_r\}$ and Theorem 2.7.

Lemma 3.3. If $r \geq 3$ and $n \geq r+1$, then $\sigma(K_{r+1} - P_2, n) \geq (r-1)(2n-r) - 2(n-r) + 2$.

Proof. By Lemma 2.2, for $r \geq 3$ and $n \geq r+1$, $\sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$. Obviously, for $r \geq 3$ and $n \geq r+1$, $\sigma(K_{r+1} - P_2, n) \geq \sigma(K_{r+1} - K_3, n) \geq (r-1)(2n-r) - 2(n-r) + 2$.

Lemma 3.4. If $r \geq 3, r+1 \geq k \geq 4$ and $n \geq r+1$, then $\sigma(K_{r+1} - H, n) \geq (r-1)(2n-r) - 2(n-r)$, for H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices.

Proof. Let

$$G = K_{r-2} + \overline{K_{n-r+2}}$$

Then G is a unique realization of $((n-1)^{r-2},(r-2)^{n-r+2})$ and G clearly does not contain $K_{r+1}-H$, where the symbol x^y means x repeats y times in the sequence. Thus

$$\sigma(K_{r+1}-H,n) \geq (r-2)(n-1)+(r-2)(n-r+2)+2 = (r-1)(2n-r)-2(n-r).$$

The Proof of Theorem 1.1 According to Lemma 3.3, it is enough to verify that for $r \geq 3$ and $n \geq 4r+8$,

$$\sigma(K_{r+1}-P_2,n) \leq (r-1)(2n-r)-2(n-r)+2.$$

We now prove that if $n \geq 4r+8$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with

$$\sigma(\pi) \geq (r-1)(2n-r)-2(n-r)+2,$$

then π is potentially $K_{r+1}-P_2$-graphic.

If $d_{r-2} \leq r-1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-3)(n-1)+(r-1)(n-r+3) \\ & = & (r-1)(n-1)-2(n-1)+(r-1)(n-r+3) \\ & = & (r-1)(2n-r)-2(n-r) \\ & < & (r-1)(2n-r)-2(n-r)+2, \end{array}$$

which is a contradiction. Thus $d_{r-2} \geq r$.

If $d_r \leq r-2$, then

$$\begin{array}{rcl} \sigma(\pi) & = & \sum_{i=1}^{r-1} d_i + \sum_{i=r}^{n} d_i \\ & \leq & (r-1)(r-2)+\sum_{i=r}^{n} min\{r-1,d_i\}+\sum_{i=r}^{n} d_i \\ & = & (r-1)(r-2)+2\sum_{i=r}^{n} d_i \\ & \leq & (r-1)(r-2)+2(n-r+1)(r-2) \\ & = & (r-1)(2n-r)-2(n-r)-2 \\ & < & (r-1)(2n-r)-2(n-r)+2, \end{array}$$

which is a contradiction. Hence $d_r \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{array}{rcl} \sigma(\pi) & = & \sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)(r-2)+\sum_{i=r}^{n} min\{r-1,d_i\}+d_r+\sum_{i=r+1}^{n} d_i \\ & = & (r-1)(r-2)+min\{r-1,d_r\}+d_r+2\sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)(r-2)+2d_r+2\sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)(r-2)+2(n-1)+2(n-r)(r-3) \\ & = & (r-1)(2n-r)-2(n-r) \\ & < & (r-1)(2n-r)-2(n-r)+2, \end{array}$$

which is a contradiction. Thus $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i=1,2,\cdots,r-2$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1}-P_2$-graphic by Lemma 3.1 or Lemma 3.2. If $d_{2r+2} \leq r-2$ and there exists an integer i,$1 \leq i \leq r-2$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned}\sigma(\pi) &\leq (i-1)(n-1)+(2r+1-i+1)(2r-i-1) \\ &\quad +(r-2)(n+1-2r-2) \\ &= i^2+i(n-4r-2)-(n-1)+(2r-1)(2r+2) \\ &\quad +(r-2)(n-2r-1).\end{aligned}$$

Since $n \geq 4r+8$, it is easy to see that $i^2+i(n-4r-2)$, consider as a function of i, attains its maximum value when $i=r-2$. Therefore,

$$\begin{aligned}\sigma(\pi) &\leq (r-2)^2+(n-4r-2)(r-2)-(n-1) \\ &\quad +(2r-1)(2r+2)+(r-2)(n-2r-1) \\ &= (r-1)(2n-r)-2(n-r)+2-n+4r+7 \\ &< \sigma(\pi),\end{aligned}$$

which is a contradiction.

Thus, $\sigma(K_{r+1}-P_2,n) \leq (r-1)(2n-r)-2(n-r)+2$ for $n \geq 4r+8$.

The Proof of Theorem 1.2 According to Lemma 3.4, it is enough to verify that for $r \geq 3$ and $n \geq 4r+10$,

$$\sigma(K_{r+1}-T_3,n) \leq (r-1)(2n-r)-2(n-r).$$

We now prove that if $n \geq 4r+10$ and $\pi=(d_1,d_2,\cdots,d_n)\epsilon GS_n$ with

$$\sigma(\pi) \geq (r-1)(2n-r)-2(n-r),$$

then π is potentially $K_{r+1}-T_3$-graphic.

If $d_{r-2} \leq r-1$, we consider the following cases.

(1)Suppose $d_{r-2}=r-1$ and $\sigma(\pi)=(r-3)(n-1)+(r-1)(n-r+3)$, then $\pi=((n-1)^{r-3},(r-1)^{n-r+3})$. Obviously π is potentially $K_{r+1}-T_3$ graphic.

(2)Suppose $d_{r-2}=r-1$ and $\sigma(\pi)<(r-3)(n-1)+(r-1)(n-r+3)$, then

$$\begin{aligned}\sigma(\pi) &< (r-3)(n-1)+(r-1)(n-r+3) \\ &= (r-1)(n-1)-2(n-1)+(r-1)(n-r+3) \\ &= (r-1)(2n-r)-2(n-r),\end{aligned}$$

which is a contradiction.

(3)Suppose $d_{r-2}<r-1$, then

$$\begin{aligned}\sigma(\pi) &< (r-3)(n-1)+(r-1)(n-r+3) \\ &= (r-1)(n-1)-2(n-1)+(r-1)(n-r+3) \\ &= (r-1)(2n-r)-2(n-r),\end{aligned}$$

which is a contradiction.

Thus, $d_{r-2} \geq r$ or π is potentially $K_{r+1} - T_3$ graphic.

If $d_r \leq r-2$, then

$$\begin{array}{rcl} \sigma(\pi) & = & \sum_{i=1}^{r-1} d_i + \sum_{i=r}^{n} d_i \\ & \leq & (r-1)(r-2) + \sum_{i=r}^{n} min\{r-1, d_i\} + \sum_{i=r}^{n} d_i \\ & = & (r-1)(r-2) + 2\sum_{i=r}^{n} d_i \\ & \leq & (r-1)(r-2) + 2(n-r+1)(r-2) \\ & = & (r-1)(2n-r) - 2(n-r) - 2 \\ & < & (r-1)(2n-r) - 2(n-r), \end{array}$$

which is a contradiction. Hence $d_r \geq r-1$.

If $d_{r+1} \leq r-3$, we consider the following cases.

(1)Suppose $d_r = n-1$, then $d_1 \geq d_2 \geq \cdots \geq d_{r-1} \geq d_r = n-1$, therefore $d_1 = d_2 = \cdots = d_r = n-1$. Therefore $d_{r+1} \geq r$, which is a contradiction.

(2)Suppose $d_r \leq n-2$, then

$$\begin{array}{rcl} \sigma(\pi) & = & \sum_{i=1}^{r-1} d_i + d_r + \sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)(r-2) + \sum_{i=r}^{n} min\{r-1, d_i\} + d_r + \sum_{i=r+1}^{n} d_i \\ & = & (r-1)(r-2) + min\{r-1, d_r\} + d_r + 2\sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)(r-2) + 2d_r + 2\sum_{i=r+1}^{n} d_i \\ & \leq & (r-1)(r-2) + 2(n-2) + 2(n-r)(r-3) \\ & = & (r-1)(2n-r) - 2(n-r) - 2 \\ & < & (r-1)(2n-r) - 2(n-r), \end{array}$$

which is a contradiction.

Thus $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \cdots, r-2$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - T_3$ graphic($\pi = ((n-1)^{r-3}, (r-1)^{n-r+3})$) or π is potentially $K_{r+1} - P_2$-graphic by Lemma 3.1 or Lemma 3.2 . Therefore, π is potentially $K_{r+1} - T_3$-graphic. If $d_{2r+2} \leq r-2$ and there exists an integer i, $1 \leq i \leq r-2$ such that $d_i \leq 2r-i-1$, then

$$\begin{array}{rcl} \sigma(\pi) & \leq & (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\ & & +(r-2)(n+1-2r-2) \\ & = & i^2 + i(n-4r-2) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1). \end{array}$$

Since $n \geq 4r+10$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a function of i, attains its maximum value when $i = r-2$. Therefore,

$$\begin{array}{rcl} \sigma(\pi) & \leq & (r-2)^2 + (n-4r-2)(r-2) - (n-1) \\ & & +(2r-1)(2r+2) + (r-2)(n-2r-1) \\ & = & (r-1)(2n-r) - 2(n-r) - n + 4r + 9 \\ & < & \sigma(\pi), \end{array}$$

which is a contradiction.

Thus, $\sigma(K_{r+1}-T_3,n) \le (r-1)(2n-r)-2(n-r)$ for $n \ge 4r+10$.

The Proof of Theorem 1.3 By Lemma 3.4, for $r \ge 3, r+1 \ge k \ge 4$ and $n \ge r+1$, $\sigma(K_{r+1}-H,n) \ge (r-1)(2n-r)-2(n-r)$. Obviously, for $r \ge 3, r+1 \ge k \ge 4$ and $n \ge 4r+10$, $\sigma(K_{r+1}-H,n) \le \sigma(K_{r+1}-T_3,n)$. By theorem 1.2, for $r \ge 3, r+1 \ge k \ge 4$ and $n \ge 4r+10$, $\sigma(K_{r+1}-T_3,n) = (r-1)(2n-r)-2(n-r)$. Then $\sigma(K_{r+1}-H,n) = (r-1)(2n-r)-2(n-r)$, for $r \ge 3, r+1 \ge k \ge 4$ and $n \ge 4r+10$.

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